Solutions, Third Annual ECC Undergraduate Mathematics Competition, April 1, 2000

1. A balance of sorts.

The sequence starts with 17 and ends with 115.

Call the first term $a+1$. Then

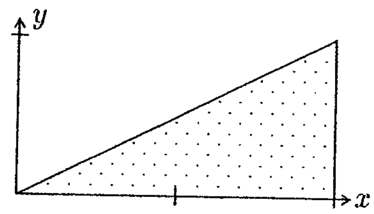
$$(a+1)+(a+2)+\cdots+(a+66)=(a+67)+(a+68)+\cdots+(a+99).$$

Upon summing these two arithmetic progressions we obtain

$$66\frac{(a+1)+(a+66)}{2}=33\frac{(a+67)+(a+99)}{2},$$

which simplifies to $2(2a+67)=2a+166$, and gives $a=16$. Thus the first term is 17 and the last is $16+99=115$.

2. A double integral.



The value is $\frac{1}{4}(e^4-1)$. To find it we need to reverse the order of integration. The region of integration is (see sketch):

$$\{(x,y): 0\le y\le 1, 2y\le x\le 2\}=\{(x,y): 0\le x\le 2, 0\le y\le \frac{x}{2}\}.$$

Then

$$\begin{aligned}\int_0^1\int_{2y}^2 e^{x^2}\,dx\,dy &= \int_0^2\int_0^{\frac{x}{2}} e^{x^2}\,dy\,dx\\ &= \int_0^2 \frac{x}{2}e^{x^2}\,dx = \frac{1}{4}e^{x^2}\Big|_0^2\\ &= \frac{1}{4}(e^4-1).\end{aligned}$$

3. A polynomial equation.

Multiplying out we get

$$x^3+6x^2+11x+6=x^3+6x^2-7x-60,$$

so $18x=-66$, and $x=-11/3$ is the only solution.

4. Matrix square roots.

Let

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The only solutions of $B^2 = A$ are $B = \pm(1/\sqrt{2})A$.

Since $A^2 = 2A$, it is clear that such B satisfy $B^2 = A$. We now show that there are no other solutions. Let

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$B^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + dc & cb + d^2 \end{pmatrix},$$

so $B^2 = A$ implies

$$a^2 + bc = 1, \tag{1}$$
$$b(a + d) = -1, \tag{2}$$
$$c(a + d) = -1, \tag{3}$$
$$\text{and} \quad cb + d^2 = 1. \tag{4}$$

Equations (1) and (4) imply that $a^2 = d^2$, and (2) shows that $a + d \neq 0$, so $a = d$. Equations (2) and (3) imply that $b = c$. Thus we have $a^2 + b^2 = 1$ and $2ab = -1$. By adding these we see that $(a + b)^2 = 0$, so $a = -b$. Thus $d = a = \pm 1/\sqrt{2}$ and $c = b = -a$, and our claim is proved.

5. Factoring in $Z[x]$.

We show that $a = 8$. We may write the other factor as

$$ax^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x - 1.$$

Multiplying this by $x^2 - x - 1$ and equating coefficients with $ax^7 + bx^6 + 1$ we obtain the equations:

$$\begin{aligned} a_4 - a &= b, \\ a_3 - a_4 - a &= 0, \\ a_2 - a_3 - a_4 &= 0, \\ a_1 - a_2 - a_3 &= 0, \\ -1 - a_1 - a_2 &= 0, \\ 1 - a_1 &= 0. \end{aligned}$$

Thus, $a_1 = 1$, and working our way upward through the system, we obtain successively, $a_2 = -2$, $a_3 = 3$, $a_4 = -5$, and $a = 8$.

SECOND SOLUTION TO PROBLEM 5

Long division leaves a remainder of $(13a+8b)x+(8a+5b+1)$. This first degree polynomial must be identically 0, so $13a+8b=0$ and $8a+5b=-1$. Solve to get $a=8$ (and $b=-13$).

6. Inverse functions.

(a) Since $f'(x)=5x^4+2>0$ for all x, f is strictly increasing, so has an inverse, g. Also, $\lim_{x\to-\infty} x^5+2x=-\infty$ and $\lim_{x\to\infty} x^5+2x=\infty$. Since f is continuous, it follows from the Intermediate Value Theorem that f maps $\boldsymbol{R}$ (the set of all real numbers) onto $\boldsymbol{R}$, and $g(f(x))=x$ for all $x\in\boldsymbol{R}$. (And, by the inverse function theorem, g is differentiable.)

(b) By the chain rule, $g'(f(x))f'(x)=1$, so

$$g'(f(x))=\frac{1}{f'(x)}.$$

Since $f(1)=3$, we have

$$g'(3)=\frac{1}{f'(1)}=\frac{1}{7}.$$

7. Area of a polygon.

(a) They intersect if and only if $r\geq\sqrt{2}$.
(b) In this case the area is given by

$$A(r)=4+2\sqrt{r^4-4}.$$

Proof: (a) Substituting $y^2=1/x^2$ into $x^2+y^2=r^2$ and clearing of fractions gives

$$x^4-r^2x^2+1=0, \tag{3}$$

which is satisfied if and only if

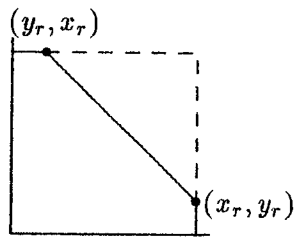
$$x^2=\frac{r^2\pm\sqrt{r^4-4}}{2}. \tag{4}$$

This has real roots if and only if $r^4\geq 4$; i.e., $r\geq\sqrt{2}$.

(b) Let x_r be the positive number such that

$$x_r^2 = \frac{r^2 + \sqrt{r^4 - 4}}{2},$$

the larger root of (3) viewed as a quadratic in x^2. Then, since the product of the roots is 1, the other root is $1/x_r^2$.



Let $y_r^2 = 1/x_r^2$. The whole configuration is symmetric with respect to both axes. The part of F_r in the positive quadrant is shown in the figure at the left, from which we also see that the area of this quarter of F_r is

$$\frac{A(r)}{4} = x_r^2 - \frac{1}{2}(x_r - y_r)^2 = x_r^2 - \frac{1}{2}(r^2 - 2),$$

since $x_r^2 + y_r^2 = r^2$ and $x_r y_r = 1$. Thus

$$\frac{A(r)}{4} = \frac{r^2 + \sqrt{r^4 - 4} - r^2 + 2}{2} = \frac{2 + \sqrt{r^4 - 4}}{2},$$

and

$$A(r) = 4 + 2\sqrt{r^4 - 4}.$$

8. Comparing two sequences.

Let $f(n) = (n+1)^n$ and $g(n) = 2^n \cdot n!$. We show that $f(n) \geq g(n)$ for all $n \geq 1$, with equality iff $n = 1$. One sees by inspection that $f(1) = g(1) = 2$, and $f(2) = 9 > g(2) = 8$. Next note that

$$\frac{g(n+1)}{g(n)} = 2n + 2,$$

while

$$\begin{aligned}
\frac{f(n+1)}{f(n)} &= \frac{(n+2)^{n+1}}{(n+1)^n} \\
&= (n+2)\left(1 + \frac{1}{n+1}\right)^n \\
&> (n+2)\left(1 + \frac{n}{n+1}\right) \\
&> n + 2 + n = 2n + 2.
\end{aligned}$$

Suppose that $f(k) > g(k)$. Then

$$\begin{aligned} f(k+1) &> (2k+2)f(k) \\ &> (2k+2)g(k) = g(k+1), \end{aligned}$$

so by induction $f(n) > g(n)$ for all $n \geq 2$.

SECOND SOLUTION

The arithmetic mean of $\{2, 4, 6, \ldots, 2n\}$ is $n+1$, and the geometric mean is $\sqrt[n]{2^n \cdot n!}$. Since the A.M $\geq$ G.M., we have $n + 1 \geq \sqrt[n]{2^n \cdot n!}$, and therefore $(n+1)^n \geq 2^n \cdot n!$, with equality if and only if all terms of the set are equal, which is the case if and only if $n = 1$.

9. Limit of a sequence.

The limit is $(\sqrt{5}-1)/2$. We show first that the limit exists by showing that the sequence is monotone increasing and bounded. First we see (by an obvious induction) that every $a_n \geq 0$. It follows that for every n,

$$0 \leq a_n = \frac{a_{n-1}+1}{a_{n-1}+2} < 1.$$

To prove monotonicity we use induction. We have $a_2 = 1/2 > a_1$. Suppose that $a_n > a_{n-1}$. Then

$$\frac{1}{a_n+2} < \frac{1}{a_{n-1}+2},$$

so that

$$a_{n+1} = \frac{a_n+1}{a_n+2} = 1 - \frac{1}{a_n+2} > 1 - \frac{1}{a_{n-1}+2} = \frac{a_{n-1}+1}{a_{n-1}+2} = a_n.$$

Thus the sequence is monotone increasing and bounded, so is convergent. Let $L = \lim_{n\to\infty} a_n$. Then taking limits on both sides in the recursion equation we find that

$$L = \frac{L+1}{L+2};$$

i.e., $L^2 + L - 1 = 0$. Since every a_n is in $[0, 1)$ we must have $0 \leq L \leq 1$. The only root of $x^2 + x - 1 = 0$ in $[0, 1]$ is $(\sqrt{5}-1)/2$, so this is the required limit.

10. Some odd integers.

Given a positive integer n, since 4^n cannot be an integral power of 10, there is an integer k such that

$$10^k < 4^n < 10^{k+1}.$$

Then $a_n = k + 1$. Now $25^n = \frac{10^{2n}}{4^n}$, and

$$10^{2n-k-1} = \frac{10^{2n}}{10^{k+1}} < \frac{10^{2n}}{4^n} < \frac{10^{2n}}{10^k} = 10^{2n-k},$$

so $b_n = 2n - k$. Thus

$$a_n + b_n = (k+1) + (2n-k) = 2n + 1,$$

an odd integer. ∎