Solutions, Fourth Annual ECC Undergraduate Mathematics Competition, March 31, 2001

1. The area is 9.

The value of m is $\sqrt[3]{54} = 3\sqrt[3]{2}$. The graphs intersect at $(0,0)$ and (m, m^2), with $x^2 < mx$ in the region between. The area is

$$\int_0^m (mx - x^2)dx = \left(\frac{mx^2}{2} - \frac{x^3}{3}\right)\Big|_0^m = \frac{m^3}{2} - \frac{m^3}{3} = \frac{m^3}{6},$$

so $\frac{m^3}{6} = 9$ and $m^3 = 54$.

2. Limit of a fraction.

The limit is $\frac{3}{2}$. Since numerator and denominator each have limit 0, we may try the Rule of L'Hôpital. The quotient of the derivatives is

$$\frac{3e^{3x}}{xf'(x) + f(x)}.$$

As $x \to 0$ this fraction approaches the limit $\frac{3}{2}$, so L'Hôpital's Rule applies and $\frac{3}{2}$ is the limit of the given fraction.

SECOND SOLUTION

$$\lim_{x\to 0} \frac{e^{3x}-1}{xf(x)} = \lim_{x\to 0} \frac{e^{3x}-1}{3x}\frac{3}{f(x)} = 1 \cdot \frac{3}{2}.$$

(Note that differentiability of f is unnecessary.)

3. Divisible by 48 for odd n.

Let $n = 2k - 1$. Then

$$\begin{aligned} F(n) &= (n^2-3)(n^2-1) \\ &= (n^2-3)(n-1)(n+1) \\ &= (4k^2-4k-2)(2k-2)(2k) \\ &= 8k(k-1)(2k^2-2k-1). \end{aligned}$$

Since one of k and $k-1$ is even, we see that $F(n)$ is divisible by 16. To see that $F(n)$ is divisible by 3, note that if k is 0 or 1 mod 3, then $k(k-1)$ is 0 mod 3, and if $k \equiv 2 \pmod 3$, then $2k^2 - 2k - 1 \equiv 0 \pmod 3$. Thus $F(n)$ is also divisible by 3, and therefore by 48.

4. Greater than or equal to 2.

We note that

$$\begin{aligned} \left|\frac{a}{b}+\frac{b}{a}\right| \geq 2 &\iff \left|\frac{a^2+b^2}{ab}\right| \geq 2 \\ &\iff a^2+b^2 \geq 2|ab| \\ &\iff |a|^2+|b|^2 \geq 2|a||b| \\ &\iff |a|^2-2|a||b|+|b|^2 \geq 0 \\ &\iff (|a|-|b|)^2 \geq 0. \end{aligned}$$

The last inequality obviously holds, and so the desired one follows.

SECOND SOLUTION

It is equivalent to prove that

$$\left|x+\frac{1}{x}\right| \geq 2 \quad \text{for all } x \neq 0.$$

Let $f(x)=|x+\frac{1}{x}|$. Then $f(-x)=f(x)$, so WLOG we assume that $x>0$ and $f(x)=x+\frac{1}{x}$. Then examination of the sign of $f'(x)=1-\frac{1}{x^2}$ shows that $f(x)$ decreases on $(0,1)$ and increases on $(1,\infty)$, so has a unique minimum at $x=1$. It follows that $f(x)\geq f(1)=2$ for all $x\neq 0$. ∎

THIRD SOLUTION

By the AM-GM inequality, $\frac{a}{b}+\frac{b}{a}\geq 2\sqrt{\frac{a}{b}\frac{b}{a}}=2$.

5. A probability.

The probability is $\frac{2072}{3125}$. The second largest is greater than 0.6 if and only if two or more of the five are greater than 0.6. The complementary event is that at most one of the five is greater than 0.6. The probability that none is greater than 0.6 is $(.6)^5$ (all five are in $(0,.6]$), and the probability that exactly one is greater than 0.6 is $\binom{5}{1}(.6)^4(.4)$. These are mutually exclusive outcomes, so the probability of their union is the sum

$$\left(\frac{3}{5}\right)^5+5\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)=\frac{3^5+10\cdot 3^4}{5^5}=\frac{3^4(13)}{5^5}=\frac{1053}{3125}.$$

Thus the probability asked for is

$$1-\frac{1053}{3125}=\frac{2072}{3125}.$$

6. Sum the series.

We will show that the sum is 1/4. Decomposing into partial fractions we obtain

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\cdot\frac{1}{n} - \frac{1}{n+1} + \frac{1}{2}\cdot\frac{1}{n+2}.$$

Then

$$\begin{aligned}\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)} &= \sum_{n=1}^{N}\left(\frac{1}{2}\frac{1}{n} - \frac{1}{n+1} + \frac{1}{2}\frac{1}{n+2}\right)\\ &= \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}\right)\\ &\quad - \left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}+\frac{1}{N+1}\right)\\ &\quad + \frac{1}{2}\left(\frac{1}{3}+\cdots+\frac{1}{N}+\frac{1}{N+1}+\frac{1}{N+2}\right).\end{aligned} \tag{1}$$

The terms

$$\frac{1}{3}+\cdots+\frac{1}{N}$$

in (1) cancel out, leaving

$$\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)} = \frac{1}{4} - \frac{1}{2}\cdot\frac{1}{N+1} + \frac{1}{2}\cdot\frac{1}{N+2}. \tag{2}$$

From (2) we see that

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)} = \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)} = \frac{1}{4}.$$

7. What is $f(-1)$?.

The answer is, $f(-1) = 1 + \tan(\frac{\pi}{4} - 1)$. To find it we solve the differential equation $y' = (x+y)^2$. Let $u = x + y$. Then

$$\frac{du}{dx} = 1 + \frac{dy}{dx} = 1 + u^2,$$

and the general solution is $\arctan u = x + c$, where c is an arbitrary constant. Then $x + y = u = \tan(x+c)$, so $f(x) = y = \tan(x+c) - x$. From $f(0) = 1$ we determine that $\tan c = 1$, and in order that f be continuous on an interval containing -1 and 0 we take $c = \frac{\pi}{4}$. Thus, $f(x) = -x + \tan(x + \frac{\pi}{4})$, and $f(-1) = 1 + \tan(\frac{\pi}{4} - 1)$.

8. Which is larger?.

We show that $a > b$. Let $x = 2001$ and $y = 2001^{2000}$ Then

$$a = \frac{y+1}{xy+1} \quad \text{and} \quad b = \frac{xy+1}{x^2y+1},$$

so that

$$\begin{aligned} a - b &= \frac{(y+1)(x^2y+1) - (xy+1)^2}{(xy+1)(x^2y+1)} \\ &= \frac{x^2y^2 + x^2y + y + 1 - x^2y^2 - 2xy - 1}{(xy+1)(x^2y+1)} \\ &= \frac{y(x-1)^2}{(xy+1)(x^2y+1)} > 0. \end{aligned}$$

SECOND SOLUTION

Write

$$a = \frac{u+1}{v+1} \quad \text{and} \quad b = \frac{v+1}{w+1},$$

where $uw = v^2$. Then

$$\frac{a}{b} = \frac{uw + u + w + 1}{v^2 + 2v + 1},$$

and since $(u+w)/2 > \sqrt{uw} = v$, we see that $a > b$.

(Looking at $a - b$ works here too.)

THIRD SOLUTION

$$1 - a = \frac{2001^{2001} - 2001^{2000}}{2001^{2001} + 1} = \frac{2001^{2000}(2001-1)}{2001^{2001}+1} = \frac{2000 \cdot 2001^{2000}}{2001^{2001}+1}.$$

$$1 - b = \frac{2001^{2002} - 2001^{2001}}{2001^{2002} + 1} = \frac{2001^{2001}(2001-1)}{2001^{2002}+1} = \frac{2000 \cdot 2001^{2001}}{2001^{2002}+1}.$$

$$\frac{1-a}{1-b} = \frac{2000 \cdot 2001^{2000}}{2001^{2001}+1} \cdot \frac{2001^{2002}+1}{2000 \cdot 2001^{2001}} = \frac{2001^{2002}+1}{(2001^{2001}+1)(2001)} = \frac{2001^{2002}+1}{2001^{2002}+2001} < 1,$$

so $1 - a < 1 - b$, and therefore $a > b$.

9. An equality among side lengths.

From the law of sines we have

$$\frac{AB}{\sin 80^\circ} = \frac{BC}{\sin 20^\circ} \quad \text{and} \quad \frac{AB}{\sin 80^\circ} = \frac{AC}{\sin 100^\circ} = \frac{CD}{\sin 40^\circ}.$$

Then

$$\begin{aligned} \overline{BC} + \overline{CD} &= \frac{\overline{AB}}{\sin 80^\circ}(\sin 20^\circ + \sin 40^\circ) \\ &= \frac{\overline{AB}}{\sin 80^\circ}\sin 20^\circ(1 + 2\cos 20^\circ). \end{aligned}$$

Now $\sin 80^\circ = 2\sin 40^\circ \cos 40^\circ = 4\sin 20^\circ \cos 20^\circ \cos 40^\circ$,so

$$\begin{aligned} \overline{BC} + \overline{CD} &= \frac{\overline{AB}(1 + 2\cos 20^\circ)}{4\cos 20^\circ \cos 40^\circ} \\ &= \frac{\overline{AB}(\cos 60^\circ + \cos 20^\circ)}{2\cos 20^\circ \cos 40^\circ}. \end{aligned}$$

But

$$\cos 60^\circ + \cos 20^\circ = 2\cos\frac{1}{2}(60^\circ + 20^\circ)\cos\frac{1}{2}(60^\circ - 20^\circ) = 2\cos 40^\circ \cos 20^\circ,$$

so we have $\overline{BC} + \overline{CD} = \overline{AB}$. ■

10. A sequence of polynomials.

From the recursion (1) we find easily that

$$P_{-1}(x) = 0,\ P_{-2}(x) = 1 = P_0(x), \quad \text{and for all } n \geq 0, P_{-n}(x) = P_{n-2}(x). \tag{3}$$

We now prove (2) for $n \geq 0$ by induction. With $n = 0$, both members are 0. Assume that

$$(P_{k-1}(x) - 1)^2 = P_k(x)P_{k-2}(x). \tag{4}$$

Then from (1) and (4) we have

$$\begin{aligned} (P_k(x) - 1)^2 - P_{k+1}(x)P_{k-1}(x) &= (P_k(x) - 1)((x^2 - 2)P_{k-1}(x) - P_{k-2}(x) + 1) \\ &\quad - ((x^2 - 2)P_k(x) - P_{k-1}(x) + 2)P_{k-1}(x) \\ &= -(x^2 - 2)P_{k-1}(x) + P_{k-2}(x) + P_k(x) - 2 \\ &= 0. \end{aligned}$$

Thus (2) holds for all $n \geq 0$. That it holds for $n < 0$ then follows from (3).