Solutions, Seventh Annual ECC Undergraduate
Mathematics Competition, April 3, 2004

1. Areas in arithmetic progression.

The radius is $\sqrt{3}$. From the arithmetic progression we have $A_2 - A_1 = (A_1 + A_2) - A_2 = A_1$. Thus $A_2 = 2A_1$ and $A_1 + A_2 = 3A_1$. Let r denote the radius of the smaller circle. Then

$$A_1 + A_2 = \pi 3^2 = 3\pi r^2,$$

so $r^2 = 3$ and $r = \sqrt{3}$.

2. Roots of quadratics.

The answer is

$$c = a + \frac{a}{b} \quad \text{and} \quad d = b + 2 + \frac{1}{b},$$

as we now show. We have $r + s = a$ and $rs = b$. Then

$$c = \left(r + \frac{1}{s}\right) + \left(s + \frac{1}{r}\right) = r + s + \frac{r+s}{rs} = a + \frac{a}{b},$$

and

$$d = \left(r + \frac{1}{s}\right)\left(s + \frac{1}{r}\right) = rs + 2 + \frac{1}{rs} = b + 2 + \frac{1}{b}.$$

3. A radical equation.

Let $u = \frac{a}{r}$. Then $a = ur,\ b = us,\ c = ut$, and

$$\sqrt{ar} + \sqrt{bs} + \sqrt{ct} = \sqrt{ur^2} + \sqrt{us^2} + \sqrt{ut^2} = (r + s + t)\sqrt{u}, \tag{1}$$

and

$$\sqrt{(a + b + c)(r + s + t)} = \sqrt{(ur + us + ut)(r + s + t)} = (r + s + t)\sqrt{u}. \tag{2}$$

From (1) and (2) we have

$$\sqrt{ar} + \sqrt{bs} + \sqrt{ct} = \sqrt{(a + b + c)(r + s + t)},$$

as desired.

4. A rational number.

Let $x = \log_{64} 27$ and $y = \log_9 32$. We will show that $xy = \frac{5}{4}$. We have $64^x = 27 = 3^3$ and $9^y = 32 = 2^5$. But $64^x = 2^{6x}$ and $9^y = 3^{2y}$, so we have

$$2^{6x} = 3^3 \quad \text{and} \quad 3^{2y} = 2^5.$$

Taking logs to any base (but the same base throughout) we have

$$6x \log 2 = 3 \log 3 \quad \text{and} \quad 2y \log 3 = 5 \log 2.$$

Multiplying we obtain

$$12xy \log 2 \log 3 = 15 \log 3 \log 2$$

so that $12xy = 15$ and $xy = \frac{5}{4}$.

5. Coin tossing.

The smallest such n is 4.

The complementary event, $X > n$, is characterized by the fact that no 6 occurs in the first n trials. This has probability $(\frac{5}{6})^n$, so

$$P(X \le n) = 1 - \left(\frac{5}{6}\right)^n \ge \frac{1}{2} \iff \left(\frac{5}{6}\right)^n \le \frac{1}{2}.$$

Now,

$$\left(\frac{5}{6}\right)^3 = \frac{125}{216} > \frac{1}{2} \quad \text{but} \quad \left(\frac{5}{6}\right)^4 = \frac{625}{1296} < \frac{1}{2},$$

so $n = 4$.

6. Which is larger?

We'll show that $h(2003) > h(2004)$.

$$h'(x) = f'(x) - xf''(x) - f'(x) = -xf''(x) < 0$$

for all $x > 0$, because $f''(x) > 0$ on $(0, \infty)$. Thus $h(x)$ is strictly decreasing on $(0, \infty)$, and $h(2003) > h(2004)$.

7. A radical sum.

The value is 22. For each n,

$$\frac{1}{\sqrt{n}+\sqrt{n+2}} = \frac{\sqrt{n+2}-\sqrt{n}}{(n+2)-n} = \frac{1}{2}(\sqrt{n+2}-\sqrt{n}),$$

so

$$\begin{aligned} S &= \frac{1}{2}\Big((\sqrt{2}-\sqrt{0}) + (\sqrt{4}-\sqrt{2}) + \cdots + (\sqrt{2004}-\sqrt{2002})\Big) \\ &= \frac{1}{2}(\sqrt{2004}-\sqrt{0}) = \sqrt{501}. \end{aligned}$$

Since $22^2 < 501 < 23^2$, $\lfloor S \rfloor = 22$.

8. Limit of a sequence.

The limit is $\frac{9}{2}\ln 3 - 2$. The sum is a Riemann sum for the integral

$$\int_1^3 x \ln x dx.$$

Integrating by parts ($u = \ln x, dv = xdx$) we get

$$\left[\frac{x^2}{2}\ln x\right]_1^3 - \int_1^3 \frac{x}{2} dx = \frac{9}{2}\ln 3 - 2.$$

9. Conditions for a solution.

Assume first that X and Y exist satisfying

$$AX + BY = C \quad \text{and} \quad BX + AY = D. \tag{1}$$

Add the two equations in (1) to obtain $(A+B)(X+Y) = C+D$. Subtracting the two equations in (1) yields $(A-B)(X-Y) = C-D$. Because $C+D$ and $C-D$ are nonsingular, it follows that $A+B$ and $A-B$ are. (For a product to be nonsingular, each factor must be nonsingular. The determinant of the product is the product of the determinants.)

Conversely, suppose that $A+B$ and $A-B$ are nonsingular. Then there exist matrices U and V such that

$$(A+B)U = C+D \quad \text{and} \quad (A-B)V = C-D. \tag{2}$$

Upon adding the two equations in (2) and dividing by 2 we obtain

$$A\Big(\frac{U+V}{2}\Big) + B\Big(\frac{U-V}{2}\Big) = C.$$

Subtracting the two equations in (2) and dividing by 2 yields

$$A\Big(\frac{U-V}{2}\Big) + B\Big(\frac{U+V}{2}\Big) = D.$$

Thus (1) holds with $X = (U+V)/2$ and $Y = (U-V)/2$.

10. An upper bound.

(a) Let

$$\begin{aligned} f(x,y) &= x^2 + y^2 \\ &= (x+y)^2 - 2xy \\ &= (z-1)^2 - 2(z^2 - 7z + 14) \\ &= -z^2 + 12z - 27 \\ &= 9 - (z-6)^2. \end{aligned}$$

This shows that $f(x,y) \leq 9$, but $z = 6$ does not correspond to real values for x and y, so the bound can be improved. The equations $x + y = u$, $xy = v$ imply $x + \frac{v}{x} = u$ and thus $x^2 - ux + v = 0$, so x is real only if $u^2 \geq 4v$. Conversely, if $u^2 \geq 4v$ there are real values for x and y. With $u = z - 1$ and $v = z^2 - 7z + 14$ the condition $u^2 \geq 4v$ translates to $z^2 - 2z + 1 \geq 4z^2 - 28z + 56$; i.e., $3z^2 - 26z + 55 \leq 0$; i.e., $(3z - 11)(z - 5) \leq 0$. This holds iff $\frac{11}{3} \leq z \leq 5$. So, the nearest z can come to 6 is with $z = 5$. Thus, $f(x,y) \leq 8$.

(b) If $z = 5$ then $x + y = 4$ and $xy = 4$, so $x = 2, y = 2, z = 5$ is a solution with $x^2 + y^2 = 8$.

SECOND SOLUTION TO PROBLEM 10 (by Lagrange multipliers)

We want to find the maximum value of $F(x,y,z) = x^2 + y^2$, subject to the conditions

$$\begin{aligned} G_1(x,y,z) &= x + y - z + 1 = 0, \quad \text{and} \\ G_2(x,y,z) &= xy - z^2 + 7z - 14 = 0. \end{aligned}$$

We put

$$F_x - \lambda G_{1x} - \mu G_{2x} = 2x - \lambda - \mu y = 0; \tag{1}$$

$$F_y - \lambda G_{1y} - \mu G_{2y} = 2y - \lambda - \mu x = 0; \tag{2}$$

$$F_z - \lambda G_{1z} - \mu G_{2z} = \lambda - \mu(-2z + 7) = 0. \tag{3}$$

Subtracting (2) from (1) gives us

$$(x - y)(\mu + 2) = 0. \tag{4}$$

Upon eliminating μ from (1) and (2) we find that

$$(x - y)(2x + 2y - \lambda) = 0. \tag{5}$$

From (4) and (5) we conclude that either (i) $x = y$, or (ii) $\mu = -2$ and $\lambda = 2x + 2y$.

(i) With $x = y$ the conditions $G_1 = G_2 = 0$ give $z = 2x + 1$ and $x^2 = z^2 - 7z + 14$. Eliminating z from these we obtain $3x^2 - 10x + 8 = 0$, with solutions $x = 2$ and $x = \frac{4}{3}$, leading to $f(x, y) = 8$ and $f(x, y) = \frac{32}{9}$, respectively.

(ii) Substituting $\mu = -2$ and $\lambda = 2x + 2y$ into (3) gives us $x + y = 2z - 7$, which together with the condition $x + y = z - 1$ implies $z = 6$. But $z = 6$ in the given conditions leads to nonreal values for x and y, so there are no solutions in case (ii).

Thus, $f(x, y) = 8$ is the maximum value, occurring when $x = y = 2$, $z = 5$.