Solutions, Ninth Annual ECC Undergraduate
Mathematics Competition, April 1, 2006

1. Roots of a quadratic.

$\boxed{s = 26.}$ If $3 + 2i$ is one root, then $3 - 2i$ is the other (because the coefficients in the equation are real). The product of the roots is $s/2$, so

$$s = 2(3 + 2i)(3 - 2i) = 2(9 + 4) = 26.$$

2. Functional equation.

$\boxed{(a, b) = (-1, 3).}$ To find these values, observe that

$$f(2) = 1 = 2a + b$$

and

$$f(-2) = 5 = -2a + b.$$

Solve this system to obtain $a = -1$ and $b = 3$.

3. Square root of an integer.

We show that $\boxed{\sqrt{5} + \sqrt{11 - 2\sqrt{30}} = \sqrt{6}.}$ Indeed,

$$(\sqrt{6} - \sqrt{5})^2 = 6 - 2\sqrt{30} + 5 = 11 - 2\sqrt{30},$$

so $\sqrt{6} - \sqrt{5} = \sqrt{11 - 2\sqrt{30}}$.

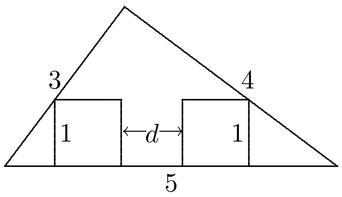
(To find this solution, one seeks an integer n such that $\sqrt{n} - \sqrt{5} = \sqrt{11 - 2\sqrt{30}}$; i.e., such that $n + 5 - 2\sqrt{5n} = 11 - 2\sqrt{30}$. Thus, one needs $n + 5 = 11$ and $5n = 30$, and $n = 6$ satisfies both conditions.)

4. Limit of a quotient.

The limit is $\boxed{e^{15}}$. Let $F(t) = \int_2^t e^{(x^4-1)}dx$. Then by the Fundamental Theorem of Calculus, $F'(t) = e^{(t^4-1)}$. Now

$$\lim_{h\to 0} \frac{\int_2^{2+h} e^{(x^4-1)}dx}{h} = \lim_{h\to 0} \frac{F(2+h) - F(2)}{h} = F'(2) = e^{15}.$$

5. Distance between squares.



We show that $\boxed{d = \frac{11}{12}}$. Because the small triangle in the lower left is similar to the (3,4,5) triangle, we know that the side of length 3 in the figure has slope $\frac{4}{3}$, so the base of the small triangle in the lower left is $\frac{3}{4}$. The side of length 4 has slope $-\frac{3}{4}$, so the base of the small triangle in the lower right is $\frac{4}{3}$. Then

$$d = 5 - \frac{3}{4} - \frac{4}{3} - 2 = \frac{11}{12}.$$

6. Which integral is larger?

$\boxed{\text{The larger is } I_1 = \int_0^1 \cos x^2 dx.}$ For, with $0 < x < 1$, we have $0 < x^2 < x < \sqrt{x} < 1$, and in the interval $(0, 1)$ the cosine function is decreasing, so

$$\cos x^2 > \cos x > \cos \sqrt{x},$$

and therefore

$$I_1 = \int_0^1 \cos x^2 dx > I_2 = \int_0^1 \cos \sqrt{x} dx.$$

7. Divisible by 33.

It suffices to show that it is divisible by 3 and by 11. Each of 5, 11 and 17 is congruent to -1 mod 3, so every odd power of each is $-1 \pmod 3$, and the sum is $-3 \equiv 0 \pmod 3$. Also,

$$17^{2n+1} \equiv (-5)^{2n+1} \equiv -5^{2n+1} \pmod{11},$$

so that $5^{2n+1} + 17^{2n+1} \equiv 0 \pmod{11}$. Of course, $11^{2n+1} \equiv 0 \pmod{11}$, so the sum is divisible by 11. ∎

Here is an alternate proof, by induction: With $n = 0$ we have 33. For the induction step, we wish to show that $5^{2n+1} + 11^{2n+1} + 17^{2n+1} = 25x + 121y + 289z$ is divisible by 33, given that $x + y + z$ is divisible by 33. Now

$$\begin{aligned} 25x + 121y + 289z &\equiv 25x + 22y + 25z \pmod{33} \\ &\equiv 25(x + y + z) - 3y \pmod{33} \\ &\equiv 0 \pmod{33} \end{aligned}$$

because 11 divides y. ∎

8. Urns and balls.

The desired probability is $\boxed{89/105}$. It is easier to calculate the complementary probability that the sum is less than or equal to 6. There are $\binom{21}{2} = 210$ different pairs of balls. Of these,

$$\binom{1}{1}\binom{2}{1} = 1 \cdot 2 = 2 \quad \text{have sum 3;}$$

$$\binom{1}{1}\binom{3}{1} + \binom{2}{2} = 1 \cdot 3 + 1 = 4 \quad \text{have sum 4;}$$

$$\binom{1}{1}\binom{4}{1} + \binom{2}{1}\binom{3}{1} = 4 + 6 = 10 \quad \text{have sum 5; and}$$

$$\binom{1}{1}\binom{5}{1} + \binom{2}{1}\binom{4}{1} + \binom{3}{2} = 5 + 8 + 3 = 16 \quad \text{have sum 6.}$$

Thus a total of 32 have sum not exceeding 6, and the remaining 178 have sum greater than 6. The desired probability is therefore $178/210 = 89/105$.

9. A sum less than 1.

$\boxed{\text{The sum is in fact } 1 - \frac{1}{\sqrt{2006}}.}$ We use the fact that

$$\frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}} = \frac{(n+1)\sqrt{n} - n\sqrt{n+1}}{(n+1)^2 n - n^2(n+1)} = \frac{(n+1)\sqrt{n} - n\sqrt{n+1}}{n(n+1)}$$
$$= \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$

to rewrite the sum

$$\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \cdots + \left(\frac{1}{\sqrt{2005}} - \frac{1}{\sqrt{2006}}\right) = 1 - \frac{1}{\sqrt{2006}}.$$

10. Differential inequalities.

If the degree of f is 1 or more, then one of $f(x) - f''(x)$ and $f'(x) - f''(x)$ has odd degree, so takes both positive and negative values. If f is constant then $f'(x) = f''(x) = 0$ for all x. Thus in every case at least one of the conditions fails for some x.