THIRD ANNUAL ECC
UNDERGRADUATE
MATHEMATICS COMPETITION

April 1, 2000

Time control: Three hours

To the team members: The problems are meant to be fun as well as challenging. You should probably not expect to work all ten of them in the allotted time. Each problem counts 10 points. Partial credit will be given for significant progress or for significant partial solutions, but a thorough job on a few of them will probably be better than some exploratory work on all.

NO BOOKS, NOTES, CALCULATORS, COMPUTERS OR NON-TEAM-MEMBERS may be consulted.

Each team may submit one solution to each problem. Think of your solution as an essay; a logical argument which makes clear why your answer to the question is correct, or why the assertion whose proof is called for in the problem is true.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be clearly given at the top of each sheet of paper.

1. A balance of sorts.

A sequence of 99 consecutive positive integers has the property that the sum of the first 66 terms is equal to the sum of the remaining 33 terms. Find the first and last terms of the sequence.

2. A double integral.

Evaluate the integral

$$\int_0^1 \int_{2y}^2 e^{x^2}\, dx dy.$$

3. A polynomial equation.

Solve

$$(x+1)(x+2)(x+3) = (x-3)(x+4)(x+5).$$

4. Matrix square roots.

Find all real 2×2 matrices which are square roots of

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

5. Factoring in $Z[x]$.

Given that a and b are integers and that $x^2 - x - 1$ is a factor of $ax^7 + bx^6 + 1$ in the ring of polynomials over the integers, determine, with proof, what the value of the integer a is.

6. Inverse functions.

Let $f(x) = x^5 + 2x$ for all real x.

(a) Show that there is a function g defined on the set of all real numbers and satisfying $g(f(x)) = x$ for all x. (You are not asked to obtain an explicit formula for g.)

(b) Find $g'(3)$.

7. Area of a polygon.

(a) For which real values of $r \geq 0$ do the curves

$$x^2 + y^2 = r^2 \qquad (1)$$

and

$$x^2y^2 = 1 \qquad (2)$$

intersect (in the real Cartesian plane)?

(b) When they intersect, let F_r be the convex polygon having these intersection points as vertices. Find the area, $A(r)$, of the polygon F_r.

8. Comparing two sequences.

Prove that for all positive integers n,

$$(n+1)^n \geq 2^n \cdot n!.$$

Under what conditions does equality occur?

9. Limit of a sequence.

The sequence $\{a_n\}$ is defined recursively by $a_1 = 0$ and for $n > 1$,

$$a_n = \frac{a_{n-1} + 1}{a_{n-1} + 2}.$$

Prove that the sequence is convergent, and find its limit. (Standard theorems from calculus or analysis may be used without proof.)

10. Some odd integers.

For each positive integer n let a_n be the number of digits in 4^n and b_n the number of digits in 25^n, both in decimal form. E.g., $a_2 = 2$ and $b_2 = 3$, since $4^2 = 16$ has 2 digits and $25^2 = 625$ has 3 digits. Show that the integer $a_n + b_n$ is always odd.