Tenth Annual ECC Undergraduate Mathematics Competition, March 31, 2007

No CALCULATORS, COMPUTERS, BOOKS, NOTES or NON-TEAM-MEMBERS may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be clearly given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit will be given for incomplete but significant work. For full credit, answers must be fully justified. (Which in some cases may simply mean showing all work and reasoning.) Have fun!

*   *   Time control: three hours   *   *

1. Triangular numbers.

A positive integer n is a *triangular number* if $n = k(k+1)/2$ for some integer k. Show that if n is triangular, so is $9n + 1$.

2. Sum of the coefficients.

If $x(x+1)(x+2)\cdots(x+2007)$ is multiplied out and like terms collected, what is the sum of the coefficients in the resulting polynomial?

3. Sum of integer parts of logs.

Is there a positive integer n such that $\sum_{k=1}^{n} \lfloor \log_3 k \rfloor = 2007$? If so, find it. If not, what is the smallest integer n such that $\sum_{k=1}^{n} \lfloor \log_3 k \rfloor > 2007$? (For real numbers x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

4. Area between circles.

Find the area between the inscribed circle and the circumscribed circle of the regular polygon of 2007 sides, if each side has length 1. Justify your answer.

5. The second largest is 9.

Seven distinct integers are chosen at random from the set $\{1, 2, 3, \ldots, 12\}$ of the first 12 positive integers. Find the probability that the second largest among those chosen is 9.

6. Equal sums.

Show that for all positive integers n,

$$\sum_{k=1}^{n}(-1)^{k+1}k^2 = (-1)^{n+1}\sum_{k=1}^{n} k.$$

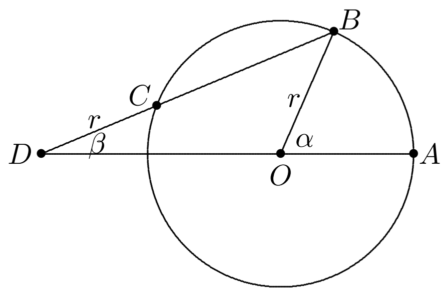
7. Sum of cosines.

Assume that θ is a number for which $\sin\theta \neq 0$. Prove that

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cdots + \cos 2007\theta = \frac{\sin 2008\theta}{2\sin\theta}.$$

8. Angle trisection.

In the figure at the right we have a circle of radius r, center O. Points A, B and C are on the circle, and D is the intersection of the extensions of lines $\overline{AO}$ and $\overline{BC}$. The length of CD is r. If, as indicated in the figure, $\angle AOB = \alpha$ and $\angle ADB = \beta$, show that $\beta = \frac{1}{3}\alpha$.



9. A limit.

Let f be a function defined in a neighborhood of 0, differentiable at 0, with $f(0) = 0$ and $f'(0) = a$. Given a positive integer m, find

$$\lim_{x\to 0}\frac{1}{x}\left(f(x) + f\left(\frac{x}{2}\right) + \cdots + f\left(\frac{x}{m}\right)\right).$$

10. At least one real root.

Find, with proof, all real numbers m such that the equation

$$z^3 + (3+i)z^2 - 3z - (m+i) = 0$$

has at least one real root.