

FIRST ANNUAL ECC
UNDERGRADUATE
MATHEMATICS COMPETITION

April 4, 1998

Time control: Three hours

To the team members: We hope you will find these problems fun as well as challenging. You should probably not expect to work all ten of them in the allotted time. Each problem counts 10 points. While partial credit will be given for significant progress or for significant partial solutions, a thorough job on a few of them will be better than some exploratory work on all, so try to pick the ones which are most appealing to you and on which you think you can make some significant progress.

NO BOOKS, NOTES, CALCULATORS, COMPUTERS OR NON-TEAM-MEMBERS may be consulted.

Each team may submit one solution to each problem. Keep in mind that a solution is an essay; a logical argument which makes clear why your answer to the question is correct, or why the assertion whose proof is called for in the problem is true.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be clearly given at the top of each sheet of paper.

1. The necklace.

A necklace has 21 diamonds. The middle one is the largest, and they taper off in value toward each end. Beginning from one end, each successive diamond is worth \$100 more than the preceding one, until the middle one is reached. Beginning from the other end, each one is worth \$150 more than its predecessor. The total value of the diamonds in the necklace is \$47150. What is the value of the middle diamond?

2. The millionth digit.

Suppose that we write down the positive integers in order:

12345678910111213141516....

Now we count digits in this sequence. For example, the 15-th digit is 2; the 20-th is 1. What is the one millionth digit?

3. Sum the series.

As we learn in first-year calculus, the sum of the series $\sum_{n=0}^{\infty} 1/n!$ is e . Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{2n^3 + n^2 - 4n - 2}{n!}.$$

4. Max and min.

Find the (exact) maximum and minimum values of the function

$$f(x) = 4x^2|x| - 21x + \frac{6}{x}, \quad (x \neq 0),$$

for $2 \leq |x + 1| \leq 4$. (Be sure to justify your answer.)

5. An inequality.

Prove that if a , b and c are positive real numbers such that $a^2 + b^2 - ab = c^2$, then

$$(a - c)(b - c) \leq 0.$$

6. Determinants

Consider the matrix

$$M = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

- (a) Find $\det(M)$, the determinant of M (and show how you obtain it).
 (b) Prove that if K is a nonsingular 5×5 matrix every element of which is 1 or -1 , then

$$|\det(K)| \geq |\det(M)|.$$

7. Not a square.

Let $P(x)$ be a polynomial with integer coefficients such that $P(1) = 2$. Prove that $P(4)$ is not the square of an integer.

8. Solutions in integers.

Find all triples of integers (a, b, n) satisfying

$$\frac{1}{a} + \frac{1}{b} = \frac{n}{a+b}.$$

9. How many regions?

The graph of a quadratic function $y = ax^2 + bx + c$ partitions the plane into two regions. Find, with proof, the maximum number of regions into which the (x, y) -plane can be divided by the graphs of 100 different quadratic functions of the form $y = ax^2 + bx + c$.

10. A functional equation.

Find all functions f defined on the nonzero real numbers and satisfying the following two conditions:

$$f\left(\frac{1}{x}\right) = xf(x) \quad \text{for all } x \neq 0, \quad (1)$$

and

$$\text{if } x \neq 0, y \neq 0 \text{ and } x + y \neq 0 \text{ then } f\left(\frac{1}{x}\right) + f\left(\frac{1}{y}\right) = 1 + f\left(\frac{1}{x+y}\right). \quad (2)$$